Reversed-Series Solution to the Universal Kepler Equation

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Introduction

PREVIOUS work[1] involving a variation of the epoch state vector using the so-called universal functions[2] has shown the need for an approximate analytic solution to the universal form of Kepler's time equation. The appealing feature of the universal function formulation is that a single expression relates orbital position and time for any orbit type, unlike the universal variable formulations,[3,4] which employ a single position variable but separate position–time relations for elliptical, parabolic, and hyperbolic trajectories. Combining these three via the universal functions avoids difficulties with differentiating the constituent relations in problems where the trajectory is changing orbital state, e.g., a low-thrust spacecraft transitioning from an elliptical to a hyperbolic path.

Our approach makes use of the universal functions as described by Battin[2] (for convenience some of the analysis is repeated here). In the universal function representation, Kepler's equation relating time and position (the universal anomaly χ) is given by

$$\chi = \alpha\sqrt{\mu}(t - t_0) + \sigma - \sigma_0 \tag{1}$$

where α is the reciprocal of the semimajor axis, μ is the gravitational parameter (universal gravitational constant times the mass of the central attracting body), t is the time, and $\sigma = \boldsymbol{r}\cdot\boldsymbol{v}/\sqrt{\mu}$, with $\boldsymbol{r}$ and $\boldsymbol{v}$ the position and velocity vectors, respectively, of the orbiting body; subscript 0 will denote evaluation at the epoch time t_0.

In Eq. (1), the quantity σ can be replaced with an expression involving only χ as a variable,

$$\sigma = \sigma_0 U_0(\chi;\alpha) + (1 - \alpha r_0)U_1(\chi;\alpha) \tag{2}$$

where the $U_i(\chi;\alpha)$ are the universal functions, defined by

$$U_n = \chi^n \sum_{i=0}^{\infty}(-1)^i \frac{(\alpha\chi^2)^i}{(n+2i)!} \tag{3}$$

To solve the Kepler problem of predicting position at any future time by using Eq. (1), one must either 1) replace the universal functions with their closed-form trigonometric equivalents (which depend on the sign of α, but this requires separate formulations for each of the three conic sections), 2) solve Eq. (1) iteratively after substituting the series representations of the U_i, or 3) invert Eq. (1) to get $\chi = \chi(t, \alpha, \sigma_0, r_0)$. This Note examines a reversed-series approximation for $\chi(t, \alpha, \sigma_0, r_0)$.

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Reversed Series

A straightforward substitution of Eqs. (2) and (3) into Eq. (1) yields a series of the form

$$(t - t_0) = b_1\chi + \frac{b_2}{2!}\chi^2 + \frac{b_3}{3!}\chi^3 + \cdots \tag{4}$$

(note that $b_0 = 0$ because $\chi = 0$ at $t = t_0$). Reversing this series leads to a solution for $\chi(t, \alpha, \sigma_0, r_0)$,

$$\chi = \sum_{i=1}^{N} \frac{c_i}{i!}(t - t_0)^i \tag{5}$$

where N is the order of the truncated series. The original series in Eq. (4) meets the necessary and sufficient conditions[5] for the inverse series to converge. Using the symbolic manipulation capability of the commercial software package Mathematica,[6] one can generate the coefficients c_i in terms of α, σ_0, r_0, and μ. This involves expressing the first two universal functions U_0 and U_1 as truncated series in χ, substituting them into Eqs. (2) and (1), and then reversing Eq. (1) to get the form of Eq. (5). The reversion can be accomplished using the internal InverseSeries routine of Mathematica or via generic algorithms described in Refs. 2 and 7 [also, Eq. (5.48) in Ref. 2 gives a specific means of generating the series for χ]. All of these methods lead to the same result.

As the order of the original truncated series is increased, the reversion process (CPU time and required memory) grows roughly as N^2 and exceeds the memory limit of Mathematica for orders greater than 15. This same restriction was found with the symbolic manipulator MAPLE.[8] Because of space limitations, only the first 10 c_i are presented here:

$$c_1 = \frac{\mu^{\frac{1}{2}}}{r_0} \qquad c_2 = \frac{-\mu\sigma_0}{r_0^3} \qquad c_3 = \frac{\mu^{\frac{3}{2}}}{r_0^3}\left(3\frac{\sigma_0^2}{r_0^2} - \frac{1}{r_0} + \alpha\right)$$

$$c_4 = \frac{\mu^2\sigma_0}{r_0^5}\left(-15\frac{\sigma_0^2}{r_0^2} + \frac{10}{r_0} - 9\alpha\right)$$

$$c_5 = \frac{\mu^{\frac{5}{2}}}{r_0^5}\left(105\frac{\sigma_0^4}{r_0^4} - 105\frac{\sigma_0^2}{r_0^3} + 90\frac{\sigma_0^2\alpha}{r_0^2} + \frac{10}{r_0^2} - 19\frac{\alpha}{r_0} + 9\alpha^2\right)$$

$$c_6 = \frac{\mu^3\sigma_0}{r_0^7}\left(-945\frac{\sigma_0^4}{r_0^4} + 1260\frac{\sigma_0^2}{r_0^3} - 1050\frac{\sigma_0^2\alpha}{r_0^2} - \frac{280}{r_0^2} + 504\frac{\alpha}{r_0} - 225\alpha^2\right)$$

$$c_7 = \frac{\mu^{\frac{7}{2}}}{r^7}\left(10{,}395\frac{\sigma_0^6}{r_0^6} - 17{,}325\frac{\sigma_0^4}{r_0^5} + 14{,}175\frac{\sigma_0^4\alpha}{r_0^4} + 6300\frac{\sigma_0^2}{r_0^4} - 10{,}962\frac{\sigma_0^2\alpha}{r_0^3} - \frac{280}{r_0^3} + 4725\frac{\sigma_0^2\alpha^2}{r_0^2} + 784\frac{\alpha}{r_0^2} - 729\frac{\alpha^2}{r_0} + 225\alpha^3\right)$$

$$c_8 = \frac{\mu^4\sigma_0}{r_0^9}\left(-135{,}135\frac{\sigma_0^6}{r_0^6} + 270{,}270\frac{\sigma_0^4}{r_0^5} - 218{,}295\frac{\sigma_0^4\alpha}{r_0^4} - 138{,}600\frac{\sigma_0^2}{r_0^4} + 235{,}620\frac{\sigma_0^2\alpha}{r_0^3} + \frac{15{,}400}{r_0^3} - 99{,}225\frac{\sigma_0^2\alpha^2}{r_0^2} - 41{,}580\frac{\alpha}{r_0^2} + 37{,}206\frac{\alpha^2}{r_0} - 11{,}025\alpha^3\right)$$

$$c_9 = \frac{\mu^{\frac{9}{2}}}{r_0^9}\left(2{,}027{,}025\frac{\sigma_0^8}{r_0^8} - 4{,}729{,}725\frac{\sigma_0^6}{r_0^7} + 3{,}783{,}780\frac{\sigma_0^6\alpha}{r_0^6}\right.$$
$$+ 3{,}153{,}150\frac{\sigma_0^4}{r_0^6} - 5{,}270{,}265\frac{\sigma_0^4\alpha}{r_0^5} - 600{,}600\frac{\sigma_0^2}{r_0^5}$$
$$+ 2{,}182{,}950\frac{\sigma_0^4\alpha^2}{r_0^4} + 1{,}580{,}040\frac{\sigma_0^2\alpha}{r_0^4} + \frac{15{,}400}{r_0^4}$$
$$- 1{,}376{,}595\frac{\sigma_0^2\alpha^2}{r_0^3} - 56{,}980\frac{\alpha}{r_0^3} + 396{,}900\frac{\sigma_0^2\alpha^3}{r_0^2}$$
$$\left. + 78{,}786\frac{\alpha^2}{r_0^2} - 48{,}231\frac{\alpha^3}{r_0} + 11{,}025\alpha^4\right)$$

$$c_{10} = \frac{\mu^5\sigma_0}{r_0^{11}}\left(-34{,}459{,}425\frac{\sigma_0^8}{r_0^8} + 91{,}891{,}800\frac{\sigma_0^6}{r_0^7}\right.$$
$$- 72{,}972{,}900\frac{\sigma_0^6\alpha}{r_0^6} - 75{,}675{,}600\frac{\sigma_0^4}{r_0^6} + 124{,}864{,}740\frac{\sigma_0^4\alpha}{r_0^5}$$
$$+ 21{,}021{,}000\frac{\sigma_0^2}{r_0^5} - 51{,}081{,}030\frac{\sigma_0^4\alpha^2}{r_0^4} - 54{,}234{,}180\frac{\sigma_0^2\alpha}{r_0^4}$$
$$- \frac{1{,}401{,}400}{r_0^4} + 46{,}332{,}000\frac{\sigma_0^2\alpha^2}{r_0^3} + 5{,}045{,}040\frac{\alpha}{r_0^3}$$
$$- 13{,}097{,}700\frac{\sigma_0^2\alpha^3}{r_0^2} - 6{,}779{,}916\frac{\alpha^2}{r_0^2}$$
$$\left. + 4{,}029{,}300\frac{\alpha^3}{r_0} - 893{,}025\alpha^4\right)$$

Results

Because of the large exponents that appear in the reversed series, it is necessary to work in a system of dimensionless units where the numerical values of μ, σ_0, and r_0 remain relatively small. For the examples given here, $\mu = 4\pi^2$ and distances are measured in Earth radii; in each case, the radius of periapsis is $r_p = 6678$ km.

Future position and velocity can be represented in terms of their initial values and the Lagrange coefficients F, G, F_t, and G_t (the same approach as used in Refs. 3 and 9 but with slightly different notation):

$$\boldsymbol{r} = F\boldsymbol{r}_0 + G\boldsymbol{v}_0 \qquad \boldsymbol{v} = F_t\boldsymbol{r}_0 + G_t\boldsymbol{v}_0 \tag{6}$$

In terms of the universal functions,

$$F = 1 - (1/r_0)U_1 \qquad G = (r_0/\sqrt{\mu})U_1 + (\sigma_0/\sqrt{\mu})U_2$$
$$F_t = [-\sqrt{\mu}/(rr_0)]U_1 \qquad G_t = 1 - (1/r)U_2 \tag{7}$$

Figure 1 depicts the maximum increase in true anomaly (from periapsis) for which the error in each of the Lagrange coefficients is less than 1%. The errors here are computed with respect to separate analytic forms of the Lagrange coefficients for each of the three conic sections.[2]

Comparisons of the reversed series solution and that of the classical power series (generated via Lagrange's fundamental invariants[2]) reveal that the reversed series converges faster, even at small eccentricities. For example, Table 1 shows the maximum error among the Lagrange coefficients for three different orbital eccentricities; each orbit has $r_p = 6678$ km, and the errors are evaluated after a single step of $\Delta f = 50$ deg from periapsis. The order of the expansion in both methods is 10.

Over large changes in true anomaly, e.g., $\Delta f = 90$ deg, the Lagrange coefficients become sensitive to errors in χ, requiring a restart or rectification procedure. After determining $\boldsymbol{r}$ and $\boldsymbol{v}$ at some time t, one uses them to compute new values of σ_0 and r_0 (with t_0 reset to t) from which to calculate the next $\boldsymbol{r}$, $\boldsymbol{v}$ pair. In this manner, one can propagate an orbit of high eccentricity ($e > 0.9$) through several revolutions while holding the errors in the Lagrange coefficients to less than 1%. Figure 1 gives some indication of the maximum value of Δf for given e (and a relatively low value of r_p) that can be used between rectifications.

Table 1 Error comparison between reversed series and Lagrange's fundamental invariants

	Maximum error in F, G, F_t, G_t	
e	Reversed series, %	Lagrange fundamental invariants, %
0.05	6.0×10^{-6}	5.3×10^{-4}
0.5	9.4×10^{-3}	0.24
5	2.3	27

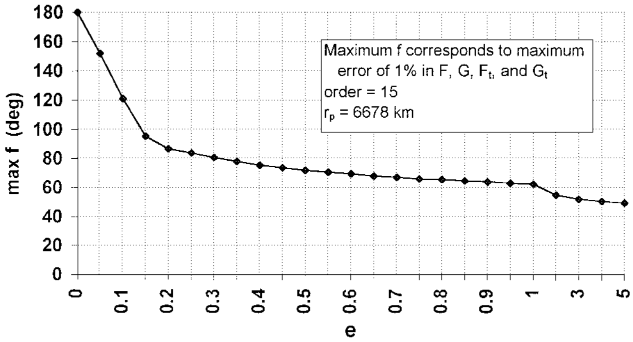

Fig. 1 Example of maximum true anomaly f vs eccentricity of the orbit.

Conclusion

The reversed series solution to the universal Kepler equation provides a useful analytic approximation that takes advantage of the singularity-free (as $e \to 1$) universal function formulation. Further, the solution is compatible with any variational approaches that use the universal functions to examine small disturbances or the effects of low-level thrust. To control error over a large change in true anomaly, a rectification step can be inserted.

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